Generalized Lagrangian of $N = 1$ supergravity

and

its canonical constraints with the real Ashtekar variable

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Abstract

We generalize the Lagrangian of $N = 1$ supergravity (SUGRA) by using an arbitrary parameter ξ, which corresponds to the inverse of Barbero's parameter β. This generalized Lagrangian involves the chiral one as a special case of the value $\xi = \pm i$. We show that the generalized Lagrangian gives the canonical formulation of $N = 1$ SUGRA with the real Ashtekar variable after the 3+1 decomposition of spacetime. This canonical formulation is also derived from those of the usual $N = 1$ SUGRA by performing Barbero's type canonical transformation with an arbitrary parameter $\beta\,(= \xi^{-1})$. We give some comments on the canonical formulation of the theory.

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1

A very simple, polynomial form of Hamiltonian constraint in canonical formulation of general relativity is obtained by using the complex Ashtekar's connection variable [1]. However, it is difficult to deal with the reality condition especially at quantum level, which must be imposed in order to select the physical, Lorentzian theory [2]. A possible way to solve this problem of the reality condition was proposed by Barbero [3] using the real-valued Ashtekar variable, although one must discard the polynomiality of the Hamiltonian constraint in the Lorentzian sector.

The advantage of the formulation with the real Ashtekar variable in pure gravity is that it provides a mathematically rigorous kinematical framework in the context of diffeomorphism invariant quantization with the Gauss and vector constraints being satisfied [4]. Furthermore, Thiemann has recently succeeded in constructing a quantum Hamiltonian constraint operator which is mathematically well-defined in the Lorentzian sector [5].

Canonical formulation of general relativity with the real Ashtekar variable has been made starting from the generalized Einstein-Cartan (EC) action [6]. [1] In this paper we extend the action to include spinor matter fields, and then derive the canonical formulation of $N = 1$ supergravity (SUGRA) with the real Ashtekar variable.

We begin with briefly reviewing the generalization of the EC Lagrangian [6]. We denote the tetrad field as e^i_μ, from which the metric field $g_{\mu\nu}$ is constructed via $g_{\mu\nu} = \eta_{ij}e^i_\mu e^j_\nu$. [2] The Lorentz connection $A_{ij\mu}$ is treated as independent variable in the EC Lagrangian. Then the generalized EC Lagrangian density, which derives

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[1] In Ref.[6] this generalized action is called the generalized Hilbert-Palatini action, which corresponds to the generalization of the tetrad form of the Palatini action.

[2] Greek letters $\mu, \nu, \cdots$ are spacetime indices, and Latin letters $i, j, \cdots$ are local Lorentz indices. We denote the Minkowski metric by $\eta_{ij} = \text{diag}(-1, +1, +1, +1)$. The totally antisymmetric tensor ϵ_{ijkl} is normalized as $\epsilon_{0123} = +1$. We define $\epsilon_{\mu\nu\rho\sigma}$ and $\epsilon^{\mu\nu\rho\sigma}$ as tensor densities which take values of $+1$ or -1.

2

Barbero's results using the real Ashtekar variable, takes the form

$$\mathcal{L}_G = \frac{e}{2} \, e_i^\mu e_j^\nu \left(R^{ij}{}_{\mu\nu} - \frac{\xi}{2} \epsilon^{ij}{}_{kl} R^{kl}{}_{\mu\nu} \right),$$ (1)

where $e = \det(e_\mu^i)$ and $R^{ij}{}_{\mu\nu}$ is the curvature tensor with respect to the Lorentz connection $A_{ij\mu}$. Here a complex parameter ξ is introduced in (1) in order to cover various types of the canonical formulation of general relativity: [3] Indeed, for $\xi = 0$, Eq.(1) is simply the EC Lagrangian which leads to the ADM canonical formulation. For $\xi = +i$ $(-i)$, only the self-dual (antiself-dual) part of the curvature contributes to the Lagrangian density (1). [4] In this case the complex (anti)self-dual connection $A_{ij\mu}^{(\pm)}$ is regarded as an independent variable and then Eq.(1) leads to Ashtekar's canonical formulation after the 3+1 decomposition of spacetime. On the other hand, the canonical formulation using the real Ashtekar variable is derived by putting ξ to be real. [5]

Such a generalization of the EC Lagrangian as given by (1) does not affect the field equation for the tetrad in the second-order formalism [6]. In order to see this, it is convenient to introduce the variable

$$B_{ij\mu} := \frac{1}{2} \left(A_{ij\mu} - \frac{\xi}{2} \epsilon_{ij}{}^{kl} A_{kl\mu} \right),$$ (2)

which reduces to $A_{ij\mu}^{(\pm)}$ for $\xi = \pm i$. Note that (2) can be solved with respect to $A_{ij\mu}$ unless $\xi = \pm i$. Since the variation of (1) with respect to $A_{ij\mu}$ can be written as

$$\delta\mathcal{L}_G = -2 \, D_\mu(e \, e_i^\mu \, e_j^\nu) \, \delta B^{ij}{}_\nu,$$ (3)

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[3] The parameter ξ is same as the parameter α of [6] and corresponds to the inverse of Barbero's parameter β as stated in [6].

[4] We denote the self-dual and antiself-dual part of a antisymmetric tensor F_{ij} as $F_{ij}^{(\pm)}$ which satisfies $(1/2)\epsilon_{ij}{}^{kl} F_{kl}^{(\pm)} = \pm i F_{ij}^{(\pm)}$.

[5] Immirzi pointed out [7] that the Barbero's parameter β $(= \xi^{-1})$ appears as a free (real) parameter in the quantum spectrum of such geometrical quantities as length, area and volume. Therefore the parameter β is also known as the Immirzi parameter.

we get the field equation for $A_{ij\mu}$, [6]

$$D_\mu(e \ e^\mu_{[i} \ e^\nu_{j]}) = 0 \tag{4}$$

with D_μ being the covariant derivative with respect to local Lorentz indices. The equation (4) is the same as that obtained from the EC Lagrangian, and can be solved to show that the $A_{ij\mu}$ is given by the Ricci rotation coefficient $A_{ij\mu}(e)$. Thus, in the second-order formalism, the second term in (1) vanishes because of the Bianchi identity. This situation is just the same as in the case $\xi = \pm i$ [8], and therefore (1) is reduced to the ordinary Hilbert-Einstein Lagrangian of the tetrad form.

Let us now try to introduce a (Majorana) Rarita-Schwinger field in the manner consistent with the above generalization of the EC Lagrangian. For this purpose, following the construction of the chiral Lagrangian of matter fields [9], we add a total divergence term with an arbitrary complex parameter η to the ordinary Lagrangian of a (Majorana) Rarita-Schwinger field in flat space, and take the flat-space Lagrangian as

$$L_{RS} = L_{RS}(\text{ordinary}) + \frac{i}{4} \ \eta \ \partial_\mu(\epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\nu\gamma_\rho\psi_\sigma) = \epsilon^{\mu\nu\rho\sigma} \ \overline{\psi}_\mu\gamma_5\gamma_\rho\frac{1 - i\eta\gamma_5}{2}\partial_\sigma\psi_\nu. \tag{5}$$

Then we apply the minimal prescription for (5) replacing the ordinary derivative by the covariant derivative [7]

$$D_\mu = \partial_\mu + \frac{i}{2}A_{ij\mu}S^{ij}, \tag{6}$$

and define the generalized Lagrangian density of a (Majorana) Rarita-Schwinger field in curved space by

$$\mathcal{L}_{RS} = \epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu\gamma_5\gamma_\rho\frac{1 - i\eta\gamma_5}{2}D_\sigma\psi_\nu. \tag{7}$$

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[6] The antisymmetrization of a tensor with respect to i and j is denoted by $A_{[i...j]} := (1/2)(A_{i...j} - A_{j...i})$.

[7] In our convention the Lorentz generator $S_{ij} = (i/4)[\gamma_i, \gamma_j]$ and $\{\gamma_i, \gamma_j\} = -2\eta_{ij}$.

Notice that the right-hand side of (7) agrees with the chiral Lagrangian density of a (Majorana) Rarita-Schwinger field when $\eta = \pm i$.

We define the Lagrangian density $\mathcal{L}$ by the sum of (1) and (7),

$$\mathcal{L} := \mathcal{L}_G + \mathcal{L}_{RS}, \tag{8}$$

and we require the $\mathcal{L}$ to reduce to the Lagrangian density of the usual $N = 1$ SUGRA in the second-order formalism. Varying the $\mathcal{L}$ with respect to $B_{ij\mu}$, we obtain

$$D_\mu(e\, e^\mu_{[i}\, e^\nu_{j]}) = \frac{1 + \xi\,\eta}{1 + \xi^2}\, X_{ij}{}^\nu + \frac{\xi - \eta}{2(1 + \xi^2)}\, \epsilon_{ij}{}^{kl} X_{kl}{}^\nu, \tag{9}$$

where $X_{ij}{}^\mu$ is a tensor density defined by

$$
\begin{aligned}
X_{ij}{}^\mu &:= \frac{1}{1 + \eta^2}\left(\frac{\delta\mathcal{L}_{RS}}{\delta A^{ij}{}_\mu} + \frac{\eta}{2}\epsilon_{ij}{}^{kl}\frac{\delta\mathcal{L}_{RS}}{\delta A^{kl}{}_\mu}\right) \\
&= \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\, \overline{\psi}_\nu \gamma_5 \gamma_\rho S_{ij}\psi_\sigma.
\end{aligned} \tag{10}
$$

If we substitute the solution of (9) with respect to $A_{ij\mu}$ into $\mathcal{L}$, then its torsion part gives four-fermion contact terms, which coincide with the contact terms of the usual $N = 1$ SUGRA if we choose $\eta = \xi$: In fact, we obtain [8]

$$\mathcal{L}(\text{second order}) = \mathcal{L}_{N=1 \text{ usual SUGRA}}(\text{second order}) + \frac{i}{4}\,\xi\,\partial_\mu(\epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\nu\gamma_\rho\psi_\sigma) \tag{11}$$

by means of the (Fierz) identity $\epsilon^{\mu\nu\rho\sigma}(\overline{\psi}_\mu\gamma_i\psi_\nu)\gamma^i\psi_\rho \equiv 0$. On the other hand, if $\eta \neq \xi$, these parameters survive in the contact terms as is seen from (9). Thus we shall assume that $\eta = \xi$ henceforth.

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[8]The divergence term of (11) is just the Chern-Simons type boundary term, which generates the chiral SUGRA 'on shell' for $\xi = \pm i$ [10, 11].

The generalized Lagrangian density of $N = 1$ SUGRA in first-order form is now given by

$$\mathcal{L} = \frac{e}{2}e_i^\mu e_j^\nu \left(R^{ij}{}_{\mu\nu} - \frac{\xi}{2}\epsilon^{ij}{}_{kl}R^{kl}{}_{\mu\nu}\right) + \epsilon^{\mu\nu\rho\sigma}\,\overline{\psi}_\mu\gamma_5\gamma_\rho\frac{1 - i\xi\gamma_5}{2}D_\sigma\psi_\nu, \qquad (12)$$

which is reduced to the Lagrangian density of $N = 1$ chiral SUGRA for $\xi = \pm i$. In case of the non-chiral theory with $\xi \neq \pm i$, the Lagrangian density of (12) is invariant under the following first-order (i.e. 'off-shell') SUSY transformations generated by a Majorana spinor parameter α; namely,

$$\delta\psi_\mu = D_\mu\alpha, \qquad (13)$$

$$\delta e_\mu^i = \frac{i}{2}\overline{\alpha}\,\gamma^i\psi_\mu, \qquad (14)$$

$$\delta B_{ij\mu} = \frac{1}{2}(C_{\mu ij} - e_{\mu[i}C^m{}_{mj]}), \qquad (15)$$

where we define $C^{\lambda\mu\nu}$ as

$$C^{\lambda\mu\nu} := e^{-1}\epsilon^{\mu\nu\rho\sigma}\overline{\alpha}\,\gamma_5\gamma^\lambda\frac{1 - i\xi\gamma_5}{2}D_\rho\psi_\sigma. \qquad (16)$$

The transformations of (13) and (14) are the same as those of the usual $N = 1$ SUGRA, whereas Eq.(15) differs from the usual one, since $C^{\lambda\mu\nu}$ depends on the parameter ξ. The form of (15), however, is easily read from the usual one if we note the relation

$$\frac{1 - i\xi\gamma_5}{2}A_{ij\mu}S^{ij} = B_{ij\mu}S^{ij} \qquad (17)$$

in the covariant derivative $D_\sigma\psi_\nu$ of (12). In case of the chiral theory, however, the situation is slightly different: For example, when $\xi = +i$, Eq.(15) becomes the transformation of $A_{ij\mu}^{(+)}$, i.e. $\delta B_{ij\mu}\,|_{\xi=+i} = \delta A_{ij\mu}^{(+)}$, while $A_{ij\mu}^{(-)}$ which appears in (13) is not an independent variable but a quantity given by e_μ^i and ψ_μ [12, 13].

The generalized Lagrangian (12) allows us to construct a canonical formulation of $N = 1$ SUGRA in terms of the real Ashtekar variable. Let us derive this by means of the Legendre transform of (12) using the (3+1) decomposition of spacetime. For this purpose we assume that the topology of spacetime M is $\Sigma \times R$ for some three-manifold Σ so that a time coordinate function t is defined on M. Then the time component of the tetrad can be defined as [9]

$$e_t^i = Nn^i + N^a e_a^i. \tag{18}$$

Here n^i is the timelike unit vector orthogonal to e_{ia}, i.e. $n^i e_{ia} = 0$ and $n^i n_i = -1$, while N and N^a denote the lapse function and the shift vector, respectively. Furthermore, we give a restriction on the tetrad with the choice $n_i = (-1, 0, 0, 0)$ in order to simplify the Legendre transform of (12). Once this choice is made, e_{Ia} becomes tangent to the constant t surfaces Σ and $e_{0a} = 0$. Therefore we change the notation e_{Ia} to E_{Ia} below. We also take the spatial restriction of the totally antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ as $\epsilon^{abc} := \epsilon_t{}^{abc}$, while $\epsilon^{IJK} := \epsilon_0{}^{IJK}$.

Under the above gauge condition of the tetrad, the (3+1) decomposition of (12) yields

$$\begin{aligned}
\mathcal{L} = \ & \epsilon^{abc}\epsilon_{IJK}E_a^I(E_b^J \hat{R}^{0K}{}_{tc} - N^d E_d^J \hat{R}^{0K}{}_{bc} + \frac{N}{2}\hat{R}^{JK}{}_{bc}) \\
& - \epsilon^{abc}(\overline{\psi}_b\gamma_5\gamma_c\hat{D}_t\psi_a - \overline{\psi}_a\gamma_5\gamma_t\hat{D}_b\psi_c \\
& + \overline{\psi}_t\gamma_5\gamma_a\hat{D}_b\psi_c - \overline{\psi}_b\gamma_5\gamma_c\hat{D}_a\psi_t)
\end{aligned} \tag{19}$$

with $\gamma_t = e_t^i\gamma_i = N\gamma_0 + N^a\gamma_a$ and $\gamma_a = E_a^I\gamma_I$. In (19), $\hat{D}_\mu$ is defined by

$$\hat{D}_\mu := \frac{1 - i\xi\gamma_5}{2}D_\mu, \tag{20}$$

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[9]Latin letters $a, b, \cdots$ are the spatial part of the spacetime indices $\mu, \nu, \cdots$, and capital letters $I, J, \cdots$ denote the spatial part of the local Lorentz indices $i, j, \cdots$.

and also the quantity, $\hat{R}^{ij}{}_{\mu\nu} := (1/2)\{R^{ij}{}_{\mu\nu} - (\xi/2)\epsilon^{ij}{}_{kl}R^{kl}{}_{\mu\nu}\}$, is decomposed as

$$\hat{R}^{0K}{}_{tc} = \partial_{[t}A^{0K}{}_{c]} + A^0{}_{L[t}A^{LK}{}_{c]} + \frac{\xi}{2}\,\epsilon^{IJK}(\partial_{[t}A_{IJc]}$$

$$+ A_{I0[t}A^0{}_{Jc]} + A_{IM[t}A^M{}_{Jc]}), \tag{21}$$

$$\hat{R}^{0K}{}_{bc} = \partial_{[b}A^{0K}{}_{c]} + A^0{}_{L[b}A^{LK}{}_{c]} + \frac{\xi}{2}\,\epsilon^{IJK}(\partial_{[b}A_{IJc]}$$

$$+ A_{I0[b}A^0{}_{Jc]} + A_{IM[b}A^M{}_{Jc]}), \tag{22}$$

$$\hat{R}^{JK}{}_{bc} = \partial_{[b}A^{JK}{}_{c]} + A^J{}_{0[b}A^{0K}{}_{c]} + A^J{}_{I[b}A^{IK}{}_{c]}$$

$$+ \xi\epsilon^{IJK}(\partial_{[b}A_{0Ic]} + A_{0M[b}A^M{}_{Ic]}). \tag{23}$$

The time derivative of the connection appears only in (21) and has the form of $\partial_t\{A_0{}^K{}_c - (\xi/2)\epsilon^{IJK}A_{IJc}\}$ [6]. Thus it is convenient to introduce the following variables

$$^-B^I{}_a := A_0{}^I{}_a - \frac{\xi}{2}\,\epsilon^{IJK}A_{JKa}, \tag{24}$$

$$^+B^I{}_a := A_0{}^I{}_a + \frac{\xi}{2}\,\epsilon^{IJK}A_{JKa}, \tag{25}$$

the inverses of which are given by

$$A_{0Ia} = \frac{1}{2}(^-B_{Ia} + {}^+B_{Ia}), \tag{26}$$

$$A_{IJa} = -\frac{1}{2\xi}\,\epsilon_{IJK}(^-B^K{}_a - {}^+B^K{}_a). \tag{27}$$

Using (26) and (27), the covariant derivative $\hat{D}_a$ of (20) and the decomposition of the curvature, (21)-(23), are written in terms of $^-B^I{}_a$ and $^+B^I{}_a$ as

$$\hat{D}_a = \frac{1 - i\xi\gamma_5}{2}\left\{\partial_a + \xi^{-1}\left(\frac{1 - i\xi\gamma_5}{2}{}^-B^I{}_a - \frac{1 + i\xi\gamma_5}{2}{}^+B^I{}_a\right)\gamma_5 S_{0I}\right\}, \tag{28}$$

and

$$\hat{R}^{0K}{}_{tc} = -\frac{1}{2}\partial_t{}^-B^K{}_c + \frac{1}{2}\partial_c\left(A_0{}^K{}_t - \frac{\xi}{2}\epsilon^{IJK}A_{IJt}\right)$$

8

$$+ \epsilon^{IJK} A_{0It} \left(\frac{\xi^2 - 1}{4\xi} {}^-B_{Jc} + \frac{\xi^2 + 1}{4\xi} {}^+B_{Jc} \right) - \frac{1}{2} A^{KI}{}_t {}^-B_{Ic}, \tag{29}$$

$$\hat{R}^{0K}{}_{bc} = -\partial_{[b} {}^-B^K{}_{c]} + \frac{\xi^2 - 3}{8\xi} \epsilon^{IJK} {}^-B_{I[b} {}^-B_{Jc]} + \frac{\xi^2 + 1}{4\xi} \epsilon^{IJK} {}^-B_{I[b} {}^+B_{Jc]}$$

$$+ \frac{\xi^2 + 1}{8\xi} \epsilon^{IJK} {}^+B_{I[b} {}^+B_{Jc]}, \tag{30}$$

$$\hat{R}^{JK}{}_{bc} = \epsilon^{IJK} \partial_{[b} \left(\frac{\xi^2 - 1}{2\xi} {}^-B_{Ic]} + \frac{\xi^2 + 1}{2\xi} {}^+B_{Ic]} \right) + \frac{3\xi^2 - 1}{4\xi^2} {}^-B^J{}_{[b} {}^-B^K{}_{c]}$$

$$+ \frac{\xi^2 + 1}{2\xi^2} {}^-B^{[J}{}_{[b} {}^+B^{K]}{}_{c]} - \frac{\xi^2 + 1}{4\xi^2} {}^+B^J{}_{[b} {}^+B^K{}_{c]}. \tag{31}$$

Then we see from (29) that ${}^-B^I{}_a$ is the dynamical variable, and that the kinetic terms in (19) are given by

$$- \tilde{E}^a_I \, \partial_t {}^-B^I{}_a - \epsilon^{abc} \, \overline{\psi}_b \gamma_5 \gamma_c \frac{1 - i\xi\gamma_5}{2} \, \partial_t \psi_a, \tag{32}$$

where we have used the identity

$$\tilde{E}^a_I := E E^a_I = \frac{1}{2} \epsilon^{abc} \epsilon_{IJK} E^J_b E^K_c \tag{33}$$

with E being defined by $E = \det(E^I_a)$. On the other hand, the nondynamical variables in (19) are A_{0It}, A_{IJt} and ${}^+B^I{}_a$ in addition to the lapse function N and the shift vector N^a.

For $N = 1$ chiral SUGRA with $\xi = \pm i$, the variable ${}^+B^I{}_a$ does not appear in (28)-(31), and $(1 - i\xi\gamma_5)/2$ in (28) becomes $(1 \pm \gamma_5)/2$ which generates only the right- or left-handed spinor field. The dynamical variable ${}^-B^I{}_a$ becomes the Ashtekar's one, i.e. ${}^-B^I{}_a |_{\xi = \pm i} = {}^{\text{Ash}} A^I{}_a$.

In case of the non-chiral theory with $\xi \neq \pm i$, the constraint corresponding to the nondynamical variable ${}^+B^I{}_a$ appears in addition to the constraints obtained by

varying $\mathcal{L}$ with respect to A_{0It}, A_{IJt}; namely,

$$^+P_I{}^a := \frac{\delta\mathcal{L}}{\delta^+B^I{}_a} = 0, \tag{34}$$

$$P^{It} := \frac{\delta\mathcal{L}}{\delta A_{0It}} = 0, \tag{35}$$

$$P^{IJt} := \frac{\delta\mathcal{L}}{\delta A_{IJt}} = 0. \tag{36}$$

The spatial restriction of the Lorentz connection, A_{IJa}, is determined from only these three constraints: In order to show this, we notice that A_{IJa} is identically expressed as

$$A_{IJa} = A_{IJa}(E, \psi) + \frac{E_a^K}{e}(M_{IJK} - 2M_{K[IJ]} + 2\delta_{K[I}M_{J]}), \tag{37}$$

where we define

$$M_{ij}{}^\nu := \frac{\delta\mathcal{L}}{\delta B^{ij}{}_\nu}, \tag{38}$$

and

$$M_{IJK} := E_K^a M_{IJa}, \quad M_I := e_\nu^j M_{Ij}{}^\nu. \tag{39}$$

In (37) $A_{IJa}(E, \psi)$ denotes

$$A_{IJa}(E, \psi) := A_{IJa}(E) + \kappa_{IJa} \tag{40}$$

with $A_{IJa}(E)$ being the spatial restriction of the Ricci rotation coefficients $A_{ij\mu}(e)$, while κ_{IJa} being defined as

$$\kappa_{IJa} := \frac{i}{4}(E_I^b E_J^c E_a^K \overline{\psi}_b \gamma_K \psi_c + E_I^b \overline{\psi}_b \gamma_J \psi_a - E_J^b \overline{\psi}_b \gamma_I \psi_a), \tag{41}$$

which leads to

$$\kappa_{I[ba]} := E_{[b}^J \kappa_{IJa]} = -\frac{i}{4}\overline{\psi}_b \gamma_I \psi_a. \tag{42}$$

10

If we compare (38) with Eqs.(34)-(36), we can show

$$M_{IJK} = \frac{\xi}{\xi^2 + 1} E_a^K \epsilon^{IJM} {}^+P_M{}^a, \tag{43}$$

$$M_I = -\frac{N}{\xi^2 + 1} \left(P_I{}^t - \frac{\xi}{2} \epsilon_{IJK} P^{JKt} \right) + \frac{1}{\xi^2 + 1} N^a E_a^J \left(P_{IJ}{}^t + \xi \epsilon_{IJK} P_K{}^t \right)$$

$$+ \frac{\xi}{\xi^2 + 1} E_a^J \epsilon_{IJK} {}^+P^{Ka}. \tag{44}$$

Thus the constraints give

$$A_{IJa} = A_{IJa}(E, \psi). \tag{45}$$

By virtue of (24) and (25), the nondynamical variable ${}^+B^I{}_a$ is now expressed by using the dynamical variables as

$$^+B^I{}_a = {}^-B^I{}_a + \xi \epsilon^{IJK} A_{JKa}(E, \psi)$$

$$= {}^-B^I{}_a - 2\xi \Gamma^I{}_a, \tag{46}$$

where the $SO(3)$ spin connection, $\Gamma^I{}_a$, is given by

$$\Gamma^I{}_a := -\frac{1}{2} \epsilon^{IJK} A_{JKa}(E, \psi)$$

$$= \overset{\circ}{\Gamma}{}^I{}_a(E) - \frac{i}{8} \epsilon^{IJK} \left(E_J^b E_K^c \, \overline{\psi}_b \gamma_a \psi_c + 2 E_J^b \, \overline{\psi}_b \gamma_K \psi_a \right) \tag{47}$$

with the spatial Levi-Civita spin connection $\overset{\circ}{\Gamma}{}^I{}_a(E) = (-1/2)\epsilon^{IJK} E_{Jb} \nabla_a E_K^b$.

We shall now eliminate the nondynamical variable ${}^+B^I{}_a$ from the Lagrangian density $\mathcal{L}$ of (19). The coefficients of A_{0It} and A_{IJt}, which are denoted by P^{It} and P^{IJt}, respectively, are given by

$$P^{It} = -\partial_a \tilde{E}^{Ia} + \epsilon^{IJK} \left(\frac{\xi^2 - 1}{2\xi} {}^-B_{Ja} + \frac{\xi^2 + 1}{2\xi} {}^+B_{Ja} \right) \tilde{E}_K^a$$

$$+ i\epsilon^{abc} \, \overline{\psi}_b \gamma_5 \gamma_c S_0{}^I \frac{1 - i\xi \gamma_5}{2} \psi_a, \tag{48}$$

$$P^{IJt} = \frac{\xi}{2} \epsilon^{IJK} (\partial_a \tilde{E}_K^a + \xi^{-1} \epsilon_{KMN} {}^-B^M{}_a \tilde{E}^{Na} - \xi^{-1} \epsilon^{abc} \, \overline{\psi}_b \gamma_c S_{0K} \frac{1 - i\xi \gamma_5}{2} \psi_a), \tag{49}$$

11

and therefore they satisfy $P_I{}^t = \xi\, \epsilon_{IJK}P^{IJt}$ by virtue of (46). Then we obtain

$$A_{0It}P^{It} + A_{IJt}P^{IJt} = \xi\Lambda^I\left(^-\mathcal{D}_a\tilde{E}_I^a - \xi^{-1}\epsilon^{abc}\,\overline{\psi}_b\gamma_c S_{0I}\frac{1-i\xi\gamma_5}{2}\psi_a\right), \qquad (50)$$

where Λ^I is defined by $\Lambda^I := \xi A_{0It} + (1/2)\epsilon_{IJK}A^{JK}{}_t$, and the covariant derivative is denoted as

$$^-\mathcal{D}_a\tilde{E}_I^a := \partial_a\tilde{E}_I^a + \xi^{-1}\epsilon_{IJK}{}^-B^J{}_a\tilde{E}^{Ka}. \qquad (51)$$

As for the coefficients of N, N^a and ψ_t, they are obtained by the straightforward calculation.

Consequently, the Lagrangian density $\mathcal{L}$ is written as

$$\begin{aligned}
\mathcal{L} =\ & -\tilde{E}_I^{a\,-}\dot{B}^I{}_a - \epsilon^{abc}\,\overline{\psi}_b\gamma_5\gamma_c\frac{1-i\xi\gamma_5}{2}\dot{\psi}_a \\
& + \xi\Lambda^I\mathcal{G}_I + N^a\mathcal{V}_a + N\mathcal{H} + \psi_t\mathcal{S}
\end{aligned} \qquad (52)$$

up to boundary terms. In (52), Λ^I, N^a, N and ψ_t are Lagrange multipliers, while $\mathcal{G}_I, \mathcal{V}_a, \mathcal{H}$ and $\mathcal{S}$ are the constraints corresponding to these Lagrange multipliers, which read as follows:

$$\mathcal{G}_I :=\ ^-\mathcal{D}_a\tilde{E}_I^a - \xi^{-1}\epsilon^{abc}\,\overline{\psi}_b\gamma_c S_{0I}\frac{1-i\xi\gamma_5}{2}\psi_a\ =\ 0, \qquad (53)$$

$$\begin{aligned}
\mathcal{V}_a :=\ & 2\tilde{E}^{Ib}F_{Iab} + \epsilon^{bcd}\,\overline{\psi}_b\gamma_5\gamma_a\frac{1-i\xi\gamma_5}{2}{}^-\mathcal{D}_c\psi_d \\
& + \frac{i(1+\xi^2)}{2\xi}\epsilon^{bcd}\,\overline{\psi}_b\gamma_0\psi_c\,K_{[da]}\ =\ 0,
\end{aligned} \qquad (54)$$

$$\begin{aligned}
\mathcal{H} :=\ & E^{-1}\epsilon^{IJK}\tilde{E}_I^a\tilde{E}_J^b\left\{\xi F_{Kab} - (1+\xi^2)R_{Kab}\right\} \\
& + \epsilon^{abc}\,\overline{\psi}_a\gamma_5\gamma_0\frac{1-i\xi\gamma_5}{2}{}^-\mathcal{D}_b\psi_c \\
& + \frac{i(1+\xi^2)}{4\xi}\epsilon^{abc}\,\overline{\psi}_a\gamma_I\psi_c\,K^I{}_b\ =\ 0,
\end{aligned} \qquad (55)$$

$$\begin{aligned}
\mathcal{S} :=\ & -\epsilon^{abc}\gamma_5\gamma_a\frac{1-i\xi\gamma_5}{2}{}^-\mathcal{D}_b\psi_c + \frac{1-i\xi\gamma_5}{2}{}^-\mathcal{D}_a(\epsilon^{abc}\gamma_5\gamma_b\psi_c) \\
& + \frac{i(1+\xi^2)}{2\xi}\epsilon^{abc}\gamma_0\psi_c\,K_{ba}\ =\ 0
\end{aligned} \qquad (56)$$

12

with K_{ba} being given by $K_{ba} = E_b^I K_{Ia} := E_b^I A_{0Ia}$. In Eqs.(54)-(56) the covariant derivative ${}^-\mathcal{D}_a$ acts on ψ_b as

$$
{}^-\mathcal{D}_a\psi_b := \partial_a\psi_b + \xi^{-1}{}^-B^I{}_a\gamma_5 S_{0I}\,\psi_b,
\tag{57}
$$

and the curvature tensors are defined by

$$
R_{Iab} := \partial_{[a}\Gamma_{Ib]} + \frac{1}{2}\epsilon_{IJK}\Gamma^J{}_{[a}\Gamma^K{}_{b]},
\tag{58}
$$

$$
F_{Iab} := \partial_{[a}{}^-B_{Ib]} + \frac{1}{2}\xi^{-1}\epsilon_{IJK}{}^-B^J{}_{[a}{}^-B^K{}_{b]}.
\tag{59}
$$

Note that in the vector constraint of (54) we have omitted a term proportional to the Gauss constraint of (53).

We shall give some comments on the canonical formulation described by (52). Firstly, let us give the relation of the dynamical variable ${}^-B^I{}_a$ to Barbero's or Ashtekar's one. The dynamical variable ${}^-B^I{}_a$ is written as

$$
{}^-B^I{}_a = A_0{}^I{}_a - \frac{\xi}{2}\,\epsilon^{IJK}A_{JKa} = K^I{}_a + \xi\Gamma^I{}_a.
\tag{60}
$$

This means that the canonical formulation based on the Lagrangian density (52) is obtained from the usual $N = 1$ SUGRA by the canonical transformation (60). Therefore the ${}^-B^I{}_a$ is related to Barbero's dynamical variable, ${}^{\mathrm{Bar}}A^I{}_a = \Gamma^I{}_a + \beta K^I{}_a$, which now includes the torsion part, by

$$
{}^-B^I{}_a = \xi\,{}^{\mathrm{Bar}}A^I{}_a \quad \text{with} \quad \xi = \beta^{-1}.
\tag{61}
$$

For the chiral case with $\xi = \pm i$, the dynamical variable ${}^-B^I{}_a$ becomes Ashtekar's one; namely,

$$
{}^-B^I{}_a\,|_{\xi=\pm i} = K^I{}_a \pm i\Gamma^I{}_a = {}^{\mathrm{Ash}}A^I{}_a.
\tag{62}
$$

13

Secondly, we focus our attention on those terms proportional to $(1 + \xi^2)$ in Eqs.(54)-(56) which violate parity operation. As for those of (54) and (56), the antisymmetric part of K_{ba} is given by

$$K_{[ba]} = E^I_{[b} \kappa_{0Ia]} = -\frac{i}{4} \, \overline{\psi}_b \gamma_0 \psi_a \tag{63}$$

by virtue of the Gauss constraint (53). Then we can show that the terms proportional to $(1+\xi^2)$ are canceled by other parity-violating terms in (54) and (56). As for (55), on the other hand, if the K_{ba} is given by the 'on-shell' expression (namely, that in the second-order formalism) as a sum of the extrinsic curvature and quadratic terms of ψ_a, then we can also show that four-fermion contact terms in the last term of (55) are canceled by other parity-violating terms. In case of the chiral theory with $\xi = \pm i$, however, those terms proportional to $(1 + \xi^2)$ do not appear in the constraints, which is one of the advantages in the Ashtekar formulation of $N = 1$ SUGRA.

The final comment is concerned with the dynamical variable ψ_a and its conjugate momentum. From (52) the conjugate momentum of ψ_a is given by

$$\pi^a := \frac{\delta \mathcal{L}}{\delta \dot{\psi}_a} = -\epsilon^{abc} \overline{\psi}_b \gamma_5 \gamma_c \frac{1 - i\xi\gamma_5}{2}. \tag{64}$$

However, Eq.(64) leads to the second-class constraint

$$\lambda^a := \pi^a + \epsilon^{abc} \overline{\psi}_b \gamma_5 \gamma_c \frac{1 - i\xi\gamma_5}{2} = 0 \tag{65}$$

unless $\xi = \pm i$. Therefore, in case of the non-chiral theory, we must compute the Dirac brackets among the basic field variables in order to eliminate λ^a as in the usual $N = 1$ SUGRA [15, 16]. Moreover, if we try to make the Dirac bracket of $(^-B^I{}_a, \psi_b)$, or of $(^-B^I{}_a, ^-B^J{}_b)$ vanish, we will have to change the form of $^-B^I{}_a$.

Recently, Thiemann has proposed the Lorentzian Hamiltonian constraint of spin-1/2 fields, in which the fermions is treated as half-densities [17]. For spin-3/2 fields, a similar approach is to take the weighted tetrad component, $\phi_I := E^{1/2} E_I^a \, \psi_a$, as a basic field variable [18]. As preliminaly, the canonical transformation

$$^{-} \overset{\circ}{B}{}^I{}_a := K^I{}_a + \xi \, \overset{\circ}{\Gamma}{}^I{}_a(E) \tag{66}$$

has been considered [19], where $\overset{\circ}{\Gamma}{}^I{}_a(E)$ is the spatial Levi-Civita spin connection defined in (47). Contratry to the case of spin-1/2 fields, however, it has been found that the form of $^{-} \overset{\circ}{B}{}^I{}_a$ must be changed in order to make the Dirac bracket of $(^{-} \overset{\circ}{B}{}^I{}_a, \psi_b)$, or of $(^{-} \overset{\circ}{B}{}^I{}_a, {}^{-} \overset{\circ}{B}{}^J{}_b)$ vanish.

To summarize, in this paper we have generalized the Lagrangian of $N = 1$ SUGRA by using an arbitrary parameter ξ as the extension of the pure-gravity case [6]. This generalized Lagrangian gives the canonical formulation with the real Ashtekar variable after the 3+1 decomposition of spacetime. The constraints in this formulation are also derived from those of the usual $N = 1$ SUGRA by performing Barbero's type canonical transformation with an arbitrary parameter $\beta \, (= \xi^{-1})$. In particular, for $\xi = \pm i$, the formulation of this paper is equivalent with the chiral one [12]. The detailed analysis for canonical quantization of $N = 1$ SUGRA with the real Ashtekar variable needs future investigation.

Acknowledgments

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